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On this post we explain what the adjoint of a matrix is and how to find it. Also, you will see several examples of adjoint of matrices and, finally, all the properties of this type of matrix.The definition of adjoint of a matrix is as follows:The adjoint of a matrix, also known as adjugate matrix, is the transpose of its cofactor matrix.The adjoint of a matrix is
also called classical adjoint of a matrix or adjunct matrix. Moreover, the adjoint of a matrix is denoted by adj(A).Once we've seen the meaning of the adjoint of a matrix, let's see how to calculate it:To find the adjoint of a matrix, first replace each element in the matrix by its cofactor and then transpose the matrix.Remember that the formula to
compute the i, j cofactor of a matrix is as follows:Where Mij is the i, j minor of the matrix, that is, the determinant that results from deleting the i-th row and the j-th column of the matrix.Note that the adjoint of a matrix can only be found for square matrices.Having seen the theory of the adjoint of a matrix, here are some solved examples of the
calculation of the adjoint of a matrix. We will first see the adjoint of a 2×2 dimension matrix, and then the adjoint of a 3×3 dimension matrix.Let A be the following square matrix of order 2:To compute the adjoint of matrix A, we first have to find the cofactor of each entry of the matrix. To do this, we have to apply the following formula:Now we replace
each element of matrix A by its cofactor to find the cofactor matrix of A:And finally, we simply have to transpose the cofactor matrix:On the other hand, there is a formula to find the adjoint of a 2×2 matrix without doing any calculations:However, this formula is only valid for 2×2 matrices. You can verify the formula by calculating with it the example
seen above.Let B be the following square matrix of order 3:To compute the adjoint of the 3×3 matrix we have to apply the same procedure. So, first we find the cofactor of each element of the matrix:Secondly, we replace each element of matrix B by its cofactor to determine the cofactor matrix of B:And finally, we transpose the cofactor matrix to find
the adjoint of matrix B:There is no formula to directly find the adjoint of a 3×3 matrix.➤ Did you know that the adjugate matrix is used to calculate the inverse of a matrix? See how to find the matrix inverse.The adjoint of a matrix has the following characteristics:The adjoint of the zero matrix (or null matrix) results in the zero matrix:Likewise, the
adjoint of the identity matrix of any order results in the identity matrix (of the same order).Transposing a matrix first and then finding its adjoint is the same as first finding the adjoint of the matrix and then transposing the result.The determinant of the adjoint of a matrix equals to the determinant of the matrix raised to n-1, where n is the order of the
matrix.If matrix A is invertible, then the adjoint of matrix A is equal to the product of the determinant of matrix A and the inverse of matrix A.If a matrix is invertible, then we can find the inverse of the adjoint of the matrix using the following formula:In addition, if a matrix is invertible, calculating the inverse of the matrix first and then its adjoint is
the same as calculating its adjoint first and then inverting the matrix.The adjoint of a matrix multiplication equals to the product of the adjoint of each matrix but multiplied in different order:The adjoint of a scalar multiplication is equal to the product of the scalar raised to n-1 and the adjoint of the matrix, where n is the order of the matrix.Let A be a
square matrix of order n, if the rank of matrix A is less than or equal to n-2, then the adjoint of matrix A results in 0.Let A be a square matrix of order n, if the rank of matrix A is n-1, then the rank of the adjoint of matrix A is 1. Flexi Says: The adjoint of a @$\begin{align*}3\times3\end{align*}@$ matrix is found by taking the transpose of the cofactor
matrix of the given matrix. Here are the steps to find the adjoint of a@$\begin{align*}3\times3\end{align*}@$ matrix: Step1: Find the matrix of minors for the given matrix. Step2: Apply the checkerboard of plus and minus signs to the matrix of minors to get the cofactor matrix. This is also known as the matrix of cofactors. Step3: Transpose the
matrix of cofactors to get the adjoint of the original matrix. Let's consider a @$\begin{align*}3\times3\end{align*}@$matrix @$\begin{align*}A:\end{align*}@$ @$$\begin{align*} A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \end{align*}@$$ The matrix of minors for A is: @$$\begin{align*} M = \begin{bmatrix} ei - fh &
di - fg & dh - eg \\ bi - ch & ai - cg & ah - bg \\ bf - ce & af - cd & ae - bd \end{bmatrix} \end{align*}@$$ Applying the checkerboard of plus and minus signs, we get the cofactor matrix C: @$$\begin{align*} C = \begin{bmatrix} ei - fh & fg - di & dh - eg \\ ch - bi & ai - cg & bg - ah \\ bf - ce & cd - af & ae - bd \end{bmatrix} \end{align*}@$$ Finally, the
adjoint of A is the transpose of the cofactor matrix C: @$$\begin{align*} adj(A) = C^T = \begin{bmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix} \end{align*}@$$Was this helpful?Analogy / Example Matrices are fundamental in mathematics and are crucial in various branches such as linear
algebra, computer science, and physics. One important concept derived from matrices is the determinant, which is closely related to the adjoint of a matrix.The adjoint (or adjugate) of a matrix is the transpose matrix of the cofactor of the given matrix. For any square matrix A to calculate its adjoint matrix we have to first calculate the cofactor matrix
of the given matrix and then find its determinant. To calculate the adjoint of a matrix, follow the following steps:Step 1: Calculate the Minor of all the elements of the given matrix A.Step 2: Find the Cofactor matrix C using the minor elements.Step 3: Find the Adjoint matrix of A by taking the transpose of the cofactor matrix C.For any 2×2 matrix A,
the image of its Adjoint is shown below,Minor of a MatrixThe minor of an element aij in a matrix is calculated by removing the row and column containing that element, and then finding the determinant of the remaining submatrix.Steps to Find the Minor:Step 1: Select the element aij whose minor you want to find.Step 2: Remove the row and column
containing aij.Step 3: Calculate the determinant of the resulting submatrix. This is the minor Mij.For a 2×2 matrix, the minor is simply the determinant of the 1×1 matrix that remains after removing the element's row and column.Learn more about Minors and Cofactors.,Cofactor of a MatrixThe Cofactor is the number we get when we remove the
column and row of a designated element in a matrix. It means to take one element from a matrix and delete the entire row and column of that element from the matrix; which elements present in that matrix that is called the cofactor.How to Find the Cofactor of a MatrixTo find the cofactor of an element of a matrix, we can use the following steps:Step
1: Delete the entire row and column that contains element under consideration. Step 2: Take the remaining elements as it is in the matrix after Step 1.Step 3: Find the determinant of the matrix formed in Step 2 which is called the minor of the element.Step 4: Now use the formula for the cofactor of element aij i.e., (-1)i+j Mij where Mij is the minor of
the element in the ith row and jth column which is already calculated in Step 3.Step 5: Result of Step 4 is the cofactor of the element under consideration, and similarly, we can calculate the cofactor of each element of the matrix to find the cofactor matrix of the given matrix.Example: Find the Cofactor Matrix of\bold{A =\begin{bmatrix} 1 & 2 & 3\\ 7
& 4 & 5 \\ 6 & 8 & 9 \end{bmatrix}}   .Solution:Given matrix is A =\begin{bmatrix} 1 & 2 & 3\\ 7 & 4 & 5 \\ 6 & 8 & 9 \end{bmatrix}Let's find the cofactor of element in first row third column i.e., 3.Step 1: Delete the entire row and column that contains element under consideration. i.e., \begin{bmatrix} \sout{1} & \sout{2} & \sout{3}\\ 7 & 4 &
\sout{5} \\ 6 & 8 & \sout{9} \end{bmatrix}Step 2: Take the remaining elements as it is in the matrix after Step 1.i.e., \begin{bmatrix} 7 & 4 \\ 6 & 8 \end{bmatrix}Step 3: Find the determinant of the matrix formed in Step 2 which is called the minor of the element.Minor of 3 in A = \begin{vmatrix} 7 & 4 \\ 6 & 8 \end{vmatrix} = 56 - 24 = 32Step 4:
Now use the formula for the cofactor of element aij i.e., (-1)i+j Mij Cofactor of element 3 = (-1)1+3(32) = 32Step 5: Continue the procedure for all the elements to find the cofactor matrix of A,i.e., Cofactor Matrix of A = \begin{bmatrix} -4&-33&32\\ 6&-9&4\\-2&16&-10 \end{bmatrix}Transpose of a MatrixTranspose of a matrix is the matrix that is
formed by changing the rows and columns of the matrix with each other. The transpose of the matrix A is denoted as AT or A'. If the order of the matrix A is m×n, then the order of the transpose matrix is n×m.Learn more about, Transpose of a MatrixHow to find the Adjoint of a Matrix?To find the Adjoint of a Matrix, first, we have to find the Cofactor
of each element, and then take 2 more steps. See below the steps,Step 1: Find the Cofactor of each element present in the matrix.Step 2: Create another matrix with the cofactors as its elements.Step 3: Now find the transpose of the matrix which comes from after Step 2.How to find the Adjoint of a 2×2 MatrixLet's consider an example for
understanding the method to find the adjoint of a 2×2 Matrix.Example: Find the Adjoint of\bold{\text{A} =\begin{bmatrix}2&3\\ 4&5 \end{bmatrix}}   .Solution:Given matrix is \text{A} =\begin{bmatrix}2&3\\ 4&5 \end{bmatrix}Step 1: Find the Cofactor of each element.Cofactor of element at A[1, 1]: 5Cofactor of element at A[1, 2]: -4Cofactor of
element at A[2, 1]: -3Cofactor of element at A[2, 2]: 2Step 2: Create matrix from Cofactorsi.e.,\bold{\begin{bmatrix}5&-4\\ -3&2 \end{bmatrix}}Step 3: Transpose of Cofactor matrix,\bold{Adj(A) = \begin{bmatrix}5&-3\\ -4&2 \end{bmatrix}}How to find the Adjoint of a 3×3 MatrixLet's take an example of a 3×3 Matrix to understand how to calculate
the Adjoint of that matrix.Example: Find the Adjoint of\bold{A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ \end{bmatrix}}   .Solution:A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ \end{bmatrix}Step 1: Find the Cofactor of each element.C11 = (-1)1+1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = (45 - 48) = -3C12 =
(-1)1+2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -(36 - 42) = 6C13 = (-1)1+3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = (32 - 35) = -3C21 = (-1)2+1\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = -(18 - 24) = 6C22= (-1)2+2\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = (9 - 21) = -12C23 = (-1)2+3\begin{vmatrix} 1 & 2 \\ 7 & 8
\end{vmatrix} = -(8 - 14) = 6C31= (-1)3+1\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = (12 - 15) = -3C32 = (-1)3+2\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -(6 - 12) = 6C33 = (-1)3+3\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = (5 - 8) = -3Step 2: Create matrix from CofactorsC = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3
\\ \end{bmatrix}Step 3: Transpose of Matrix C to adjoint of given matrix, rows become columns and columns become row.\operatorname{adj}(A) = C^{T}= \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \\ \end{bmatrix}Which is adjoint of given matrix A.Properties of the Adjoint of a MatrixAdjoints of a matrix have various properties, some
of which are as follows:A(Adj A) = (Adj A)A = |A| InAdj(BA) = (Adj B) (Adj A)|Adj A| = |A|n-1Adj(kA) = kn-1(Adj A)Finding the Inverse Using the Adjoint of a MatrixFinding the inverse is one of the important applications of the Adjoint of the Matrix. To find the inverse of a Matrix using the Adjoint, we can use the following steps:Step 1: Find the
determinant of the matrix.Step 2: If the determinant is zero, then the matrix is not invertible, and there is no inverse.Step 3: If the determinant is non-zero, then find the adjoint of the matrix.Step 4: Divide the adjoint of the matrix by the determinant of a matrix.Step 5: The result of Step 4 is the Inverse of the given Matrix.Example: Find the inverse
of\bold{A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ \end{bmatrix}}   .Solution:Given matrix A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ \end{bmatrix}|A| = 1(45 - 48) -2(36 - 42) + 3(32 - 35)⇒ |A| = -3 -2(-6) + 3(-3)⇒ |A| = -3 + 12 - 9 = 0Thus, inverse of A doesn't exist.Learn more about a MatrixSolved Examples of the
Adjoint of a MatrixExample 1: Find the Adjoint of the given matrixA =\begin{bmatrix} 1 & 2 & 3\\ 7 & 4 & 5 \\ 6 & 8 & 9 \end{bmatrix}   .Solution:Step 1: To find the cofactor of each elementTo find the cofactor of each element, we have to delete the row and column of each element one by one and take the present elements after
deleting.Cofactor of elements at A[1, 1] = 1: +\begin{bmatrix} 4 & 5 \\ 8 & 9 \end{bmatrix}   = +(4 x 9 - 8 x 5) = -4Cofactor of elements at A[1, 2] = 2: -\begin{bmatrix} 7 & 5 \\ 6 & 9 \end{bmatrix}   = -(7 x 9 - 6 x 5) = -33Cofactor of elements at A[1, 3] = 3: +\begin{bmatrix} 7 & 4 \\ 6 & 8 \end{bmatrix}   = +(7 x 8 - 6 x 4) = 32Cofactor
of elements at A[2, 1] = 7: -\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}   = -(2 x 9 - 8 x 3) = 6Cofactor of elements at A[2, 2] = 4: +\begin{bmatrix} 1 & 3 \\ 6 & 9 \end{bmatrix}   = +(1 x 9 - 6 x 3) = -9Cofactor of elements at A[2, 3] = 5: -\begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix}   = -(1 x 8 - 6 x 2) = 4Cofactor of elements at A[3, 1]
= 6: +\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}   = +(2 x 5 - 4 x 3) = -2Cofactor of elements at A[3, 2] = 8: -\begin{bmatrix} 1 & 3 \\ 7 & 5 \end{bmatrix}   = -(1 x 5 - 7 x 3) = 16Cofactor of elements at A[3, 3] = 9: +\begin{bmatrix} 1 & 2 \\ 7 & 4 \end{bmatrix}   = +(1 x 4 - 7 x 2) = -10The matrix looks like with the cofactors:A
=\begin{bmatrix} +\begin{bmatrix} 4 & 5 \\ 8 & 9 \end{bmatrix} & -\begin{bmatrix} 7 & 5 \\ 6 & 9 \end{bmatrix} & +\begin{bmatrix} 7 & 4 \\ 6 & 8 \end{bmatrix}\\ \\ -\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} & +\begin{bmatrix} 1 & 3 \\ 6 & 9 \end{bmatrix} & -\begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} \\ \\   +\begin{bmatrix} 2 & 3 \\
4 & 5 \end{bmatrix} & -\begin{bmatrix} 1 & 3 \\ 7 & 5 \end{bmatrix} & +\begin{bmatrix} 1 & 2 \\ 7 & 4 \end{bmatrix} \end{bmatrix}The final cofactor matrix:A =\begin{bmatrix} -4 & -33 & 32\\ 6 & -9 & 4 \\ -2 & 16 & -10 \end{bmatrix}Step 2: Find the transpose of the matrix obtained in step 1adj(A) =\begin{bmatrix} -4 & 6 & -2\\ -33 & -9 & 16 \\
32 & 4 & -10 \end{bmatrix}This is the Adjoint of the matrix.Example 2: Find the Adjoint of the given matrixA =\begin{bmatrix} -1 & -2 & -2\\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}   .Solution:Step 1: To find the cofactor of each elementTo find the cofactor of each element, we have to delete the row and column of each element one by one and
take the present elements after deleting.Cofactor of element at A[1, 1] = -1: +\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}   = +(1 x 1 - (-2) x (-2)) = -3Cofactor of elements at A[1, 2] = -2: -\begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix}   = -(2 x 1 - 2 x (-2)) = -6Cofactor of elements at A[1, 3] = -2: +\begin{bmatrix} 2 & 1 \\ 2 & -2
\end{bmatrix}   = +(2 x (-2) - 2 x 1) = -6Cofactor of elements at A[2, 1] = 2: -\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}   = -((-2) x 1 - (-2) x (-2)) = 6Cofactor of elements at A[2, 2] = 1: +\begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}   = +((-1) x 1 - 2 x (-2)) = 3Cofactor of elements at A[2, 3] = -2: -\begin{bmatrix} -1 & -2 \\ 2 & -2
\end{bmatrix}   = -((-1) x (-2) - 2 x (-2)) = -6Cofactor of elements at A[3, 1] = 2: +\begin{bmatrix} -2 & -2 \\ 1 & -2 \end{bmatrix}   = +((-2) x (-2) - 1 x (-2)) = 6Cofactor of elements at A[3, 2] = -2: -\begin{bmatrix} -1 & -2 \\ 2 & -2 \end{bmatrix}   = -((-1) x (-2) - 2 x (-2)) = -6Cofactor of elements at A[3, 3] = 1: +\begin{bmatrix} -1 & -2 \\
2 & 1 \end{bmatrix}   = +((-1) x (-1)- 2 x (-2)) = 3The final cofactor matrix:A =\begin{bmatrix} -3 & -6 & -6\\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}Step 2: Find the transpose of the matrix obtained in Step 1adj(A) =\begin{bmatrix} -3 & 6 & 6\\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}This is the Adjoint of the matrix. The adjoint of a matrix is a
fundamental concept in linear algebra, closely related to the inverse of a matrix. It is a matrix composed of the cofactors of the original matrix, transposed. This concept is particularly important in solving systems of linear equations, finding inverses, and other matrix-related computations.1.0What is the Adjoint of a Matrix?The adjoint of a matrix
(sometimes called the adjugate) is the transpose of the cofactor matrix of a given square matrix. For a matrix A, the adjoint is denoted by adj(A). The cofactor matrix is created by replacing each element of A with its corresponding cofactor.2.0Adjoint of a Matrix FormulaGiven a square matrix A, the adjoint of A, denoted by adj(A), is the transpose of its
cofactor matrix.Here's the step-by-step process to compute the adjoint:Cofactor Matrix: For each element aij of the matrix A, determine its cofactor Cij. The cofactor is the determinant of the minor matrix that remains after removing the ith row and jth column, multiplied by (−1)i+j.Construct the Cofactor Matrix: Arrange all the cofactors Cij into a
matrix, called the cofactor matrix.Transpose the Cofactor Matrix: Swap the rows and columns of the cofactor matrix to obtain the adjoint of the matrix A.Mathematically, if A is a n × n matrix, then the adjoint of A, denoted by adj(A), is given by:adj(A)=Cofactor(AT)3.0Mathematical RepresentationIf A is a square matrix of order n, then the adjoint of A,
adj(A), is given by:adj(A)=C11C12...C1nC21C22...C2n............Cn1Cn2...CnnWhere Cij is the cofactor of the element aij in matrix A.4.0Adjoint of a 2 × 2 MatrixFor a 2 × 2 matrix A=[acbd] , the adjoint is computed as follows:C11 = d C12 = –c C21 = –b C22 = a Construct the Cofactor Matrix:Cofactor(A)=[d−c−ba]Transpose the Cofactor
Matrix:adj(A)=[d−b−ca]So, the adjoint of matrix A is: adj(A)=[d−b−ca]5.0Adjoint of a 3 × 3 MatrixConsider the matrix A as: A=101240356Cofactor of A11: Minor ofA11=4056=4(6)−5(0)=24Cofactor of A12: Minor ofA12=0156=0(6)−5(1)=−5Construct the Cofactor MatrixCofactor(A)=24−12−253−5−424Transpose the Cofactor Matrixadj(A)=245−4
−1232−2−54Thus, the adjoint of the matrix A is:adj(A)=245−4−12320−1546.0Cofactor and Adjoint of a MatrixThe concept of the cofactor is central to finding the adjoint of a matrix. For any square matrix A, the cofactor of an element aij (where i is the row and j is the column) is the determinant of the submatrix that remains after removing the ith
row and jth column, multiplied by (–1)i+j.7.0Importance of the Adjoint of a MatrixThe adjoint of a matrix is crucial in finding the inverse of a matrix. For a non-singular matrix A, the inverse is given by: A−1=det(A)1adj(A)Where det(A) is the determinant of A. If the determinant is non-zero, the inverse exists, and the adjoint is used to compute
it.8.0Adjoint of a Matrix ExampleExample 1: Let's find the adjoint of a 2 × 2 matrix B:B=[2513]Solution: Cofactor of B11: Cofactor B11 = 3 Cofactor of B12: Cofactor B12 = –5 Cofactor of B21: Cofactor B21 = –1 Cofactor of B22: Cofactor B22= 2 Construct the Cofactor MatrixCofactor(B)=[3−1−52]Transpose the Cofactor Matrixadj(B)=[3−5−12]So,
the adjoint of matrix B is:adj(B)=[3−5−12]Example 2: Find the Adjoint of a 3 × 3 Matrix. Given the matrix A:A=24−13121−35Solution: To find the cofactor of each element in the matrix, we need to determine the determinant of the 2 × 2 submatrices that remain after removing the respective row and column of each element.Cofactor of a11 =
2: Minor ofa11=12−35 =(1)(5)–(–3)(2) =5+6=11Cofactor C11 = 11.Cofactor of a12 = 3:Minor of a12 = 4−1−35 =(4)(5)–(–3)(–1) =20–3=17Cofactor C12 = –17 (note the sign change due to the position of the element).Cofactor of a13 = 1:Minor of a13 = 4−112=(4)(2)–(1)(–1) =8+1=9Cofactor C13 = 9.Cofactor of a21 = 4:Minor of a21 = 3215 =(3)(5)–
(1)(2) =15–2=13Cofactor C21 = –13 (note the sign change due to the position of the element).Cofactor of a22 = 1:Minor of a22 = 2−115 =(2)(5)–(1)(–1) =10+1=11Cofactor C22 = 11.Cofactor of a23 = –3:Minor of a23 = 2−132=(2)(2)–(3)(–1) =4+3=7Cofactor C23 = –7 (note the sign change due to the position of the element).Cofactor of a31 = –
1:Minor of a31 = 311−3=(3)(–3)–(1)(1) =–9–1=–10Cofactor C31 = –10.Cofactor of a32 = 2:Minor of a32 = 241−3=(2)(–3)–(1)(4) =–6–4=–10Cofactor C32 = 10 (note the sign change due to the position of the element).Cofactor of a33 = 5:Minor of a33 =2431=(2)(1)–(3)(4) =2–12=–10Cofactor C33 = –10.Construct the Cofactor MatrixNow, arrange all
the cofactors into a cofactor matrix:Cofactor(A)=11−13−10−1711109−7−10Transpose the Cofactor MatrixFinally, transpose the cofactor matrix to get the adjoint of the matrix A:adj(A)=11−179−1311−7−1010−10The adjoint of the matrix A is:adj(A)=11−179−1311−7−1010−10Example 3: Adjoint of a 2 × 2 Matrix. Consider the matrix B: B=[4276
]Solutions:Cofactor of b11 = 4:Minor of b11 = 6Cofactor C11 = 6 .Cofactor of b12 = 7:Minor of b12 = 2Cofactor C12 = –2. (since it's in an odd position).Cofactor of b21 = 2:Minor of b21 = 7Cofactor C21 = –7. (since it's in an odd position).Cofactor of b22 = 6:Minor of b22 = 4Cofactor C22 = 4.Construct the Cofactor MatrixCofactor(B)=[6−7−24
]Transpose the Cofactor Matrixadj(B)=[6−2−74]Final AnswerThe adjoint of matrix B is:adj(B)=[6−2−74]Example 4: Adjoint of a 3x3 Matrix. Consider the matrix C:C=201140356Solutions:Cofactor of c11 = 2:Minor of c11 = 4056=(4)(6)–(5)(0)=24Cofactor C11 = 24.Cofactor of c12 = 1:Minor of c12 =0156=(0)(6)–(5)(1)=–5 Cofactor C12 = 5. (since
it's in an odd position).Cofactor of c13 = 3:Minor of c13 =0140=(0)(0)–(4)(1)=–4 Cofactor C13 = –4.Cofactor of c21 = 0:Minor of c21 = 1036=(1)(6)–(3)(0)=6Cofactor C21 = –6. (since it's in an odd position).Cofactor of c22 = 4:Minor of c22 = 2136=(2)(6)–(3)(1)=12–3=9Cofactor C22 = 9.Cofactor of c23 = 5:Minor of c23 = 2110=(2)(0)–(1)(1)=–
1Cofactor C23 = 1. (since it's in an odd position).Cofactor of c31 = 1:Minor of c31 = 1435=(1)(5)–(3)(4)=5–12=–7Cofactor C31 = –7.Cofactor of c32 = 0:Minor of c32 = 2135=(2)(5)–(3)(1)=10–3=7Cofactor C32 = –7. (since it's in an odd position).Cofactor of c33 = 6:Minor of c33 = 2014=(2)(4)–(1)(0)=8Cofactor C33 = 8.Construct the Cofactor
MatrixCofactor(C)=24−6−75910−4−18Transpose the Cofactor Matrixadj(C)=245−469−1−7108Final AnswerThe adjoint of matrix C is:adj(C)=245−469−1−7108Example 5: Consider matrix D and find the adjoint of matrix D.D=[0−110]Solutions:Cofactor of d11 = 0:Minor of d11 = 0Cofactor C11 = 0.Cofactor of d12 = 1:Minor of d12 = –1Cofactor
C12 = 1. (since it's in an odd position).Cofactor of d21 = –1:Minor of d21 = 1Cofactor C21 = –1. (since it's in an odd position).Cofactor of d22 = 0:Minor of d22 = 0Cofactor C22 = 0.Construct the Cofactor MatrixCofactor(D)=[0−110]Transpose the Cofactor Matrixadj(D)=[01−10]Final AnswerThe adjoint of matrix D is: adj(D)=[01−10]9.0Practice
Problems On Adjoint of a Matrix QuestionsFind the adjoint of the following matrix: A=43−276−5210Find the adjoint of the following matrix: B=147258369Given a 2 × 2 matrix C=[acbd] , find adj(C).Calculate the adjoint of a 4 × 4 matrix D:D=1234234134124123 Download the Testbook APP & Get Pass Pro Max FREE for 7 Days10,000+ Study
NotesRealtime Doubt Support71000+ Mock TestsRankers Test Series+ more benefitsDownload App Now Here you will learn how to find adjoint of the matrix 2×2 and 3×3, cofactors and its properties with examples. Let's begin – Adjoint of the Matrix Let A = \([a_{ij}]\) be a square matrix of order n and let \(C_{ij}\) be a cofactor of \(a_{ij}\) in A.
Then the transpose of the matrix of cofactors of elements of A is called adjoint of A and is denoted by adj A. Thus, adj A = \([C{ij}]^T\) \(\implies\) \((adj A)_{ij}\) = \(C_{ji}\) = Cofactor of \(a_{ij}\) in A. If A = \(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33} \end{bmatrix}\) then, adj A = \
({\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33} \end{bmatrix}}^T\) = \(\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32}\\ C_{13} & C_{23} & C_{33} \end{bmatrix}\) where \(C_{ij}\) denotes cofactor of \(a_{ij}\) in A. How to find Cofactors and Adjoint for
2×2 Matrix : Example : Let A = \([a_{ij}]\) = \(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\) then, cofactor of \(a_{11}\) = s and cofactor of \(a_{12}\) = -r cofactor of \(a_{21}\) = -q cofactor of \(a_{22}\) = p \(\therefore\) adj A = \({\begin{bmatrix} s & -r \\ -q & p \end{bmatrix}}^T\) = \(\begin{bmatrix} s & -q \\ -r & p \end{bmatrix}\) Rule : It is
evident from this example that the adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing signs of off-diagonal elements. If A = \(\begin{bmatrix} -2 & 3 \\ -5 & 4 \end{bmatrix}\) then by the above rule, we obtain adj A \(\begin{bmatrix} 4 & -3 \\ 5 & -2 \end{bmatrix}\) How to find Cofactors
and Adjoint for 3×3 Matrix : Example : Let A = \([a_{ij}]\) = \(\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 3 \end{bmatrix}\) Let \(C_{ij}\) be cofactor of \(a_{ij}\) in A. Then the cofactors of elements of A are given by \(C_{11}\) = \(\begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix}\) = 9, \(C_{12}\) = -\(\begin{vmatrix} 2 & -3 \\ -1 & 3
\end{vmatrix}\) = -3, \(C_{13}\) = \(\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}\) = 5 \(C_{21}\) = -\(\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}\) = -1, \(C_{22}\) = \(\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}\) = 4, \(C_{23}\) = -\(\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}\) = -3 \(C_{31}\) = \(\begin{vmatrix} 1 & 1 \\ 1 & -3
\end{vmatrix}\) = -4, \(C_{32}\) = -\(\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}\) = 5, \(C_{33}\) = \(\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}\) = -1 \(\therefore\)   adj A = \({\begin{bmatrix} 9 & -3 & 5 \\ -1 & 4 & -3 \\ -4 & 5 & -1 \end{bmatrix}}^T\) = \(\begin{bmatrix} 9 & -1 & -4 \\ -3 & 4 & 5 \\ 5 & -3 & -1 \end{bmatrix}\) Note : Let
A be a square matrix of order n. Then, A (adj A) = |A| \(I_n\) = (adj A) A. Share — copy and redistribute the material in any medium or format for any purpose, even commercially. Adapt — remix, transform, and build upon the material for any purpose, even commercially. The licensor cannot revoke these freedoms as long as you follow the license
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warranties are given. The license may not give you all of the permissions necessary for your intended use. For example, other rights such as publicity, privacy, or moral rights may limit how you use the material. On this post we explain what the adjoint of a matrix is and how to find it. Also, you will see several examples of adjoint of matrices and,
finally, all the properties of this type of matrix.The definition of adjoint of a matrix is as follows:The adjoint of a matrix, also known as adjugate matrix, is the transpose of its cofactor matrix.The adjoint of a matrix is also called classical adjoint of a matrix or adjunct matrix. Moreover, the adjoint of a matrix is denoted by adj(A).Once we've seen the
meaning of the adjoint of a matrix, let's see how to calculate it:To find the adjoint of a matrix, first replace each element in the matrix by its cofactor and then transpose the matrix.Remember that the formula to compute the i, j cofactor of a matrix is as follows:Where Mij is the i, j minor of the matrix, that is, the determinant that results from deleting
the i-th row and the j-th column of the matrix.Note that the adjoint of a matrix can only be found for square matrices.Having seen the theory of the adjoint of a matrix, here are some solved examples of the calculation of the adjoint of a matrix. We will first see the adjoint of a 2×2 dimension matrix, and then the adjoint of a 3×3 dimension matrix.Let A
be the following square matrix of order 2:To compute the adjoint of matrix A, we first have to find the cofactor of each entry of the matrix. To do this, we have to apply the following formula:Now we replace each element of matrix A by its cofactor to find the cofactor matrix of A:And finally, we simply have to transpose the cofactor matrix:On the other
hand, there is a formula to find the adjoint of a 2×2 matrix without doing any calculations:However, this formula is only valid for 2×2 matrices. You can verify the formula by calculating with it the example seen above.Let B be the following square matrix of order 3:To compute the adjoint of the 3×3 matrix we have to apply the same procedure. So,
first we find the cofactor of each element of the matrix:Secondly, we replace each element of matrix B by its cofactor to determine the cofactor matrix of B:And finally, we transpose the cofactor matrix to find the adjoint of matrix B:There is no formula to directly find the adjoint of a 3×3 matrix.➤ Did you know that the adjugate matrix is used to
calculate the inverse of a matrix? See how to find the matrix inverse.The adjoint of a matrix has the following characteristics:The adjoint of the zero matrix (or null matrix) results in the zero matrix:Likewise, the adjoint of the identity matrix of any order results in the identity matrix (of the same order).Transposing a matrix first and then finding its
adjoint is the same as first finding the adjoint of the matrix and then transposing the result.The determinant of the adjoint of a matrix equals to the determinant of the matrix raised to n-1, where n is the order of the matrix.If matrix A is invertible, then the adjoint of matrix A is equal to the product of the determinant of matrix A and the inverse of
matrix A.If a matrix is invertible, then we can find the inverse of the adjoint of the matrix using the following formula:In addition, if a matrix is invertible, calculating the inverse of the matrix first and then its adjoint is the same as calculating its adjoint first and then inverting the matrix.The adjoint of a matrix multiplication equals to the product of the
adjoint of each matrix but multiplied in different order:The adjoint of a scalar multiplication is equal to the product of the scalar raised to n-1 and the adjoint of the matrix, where n is the order of the matrix.Let A be a square matrix of order n, if the rank of matrix A is less than or equal to n-2, then the adjoint of matrix A results in 0.Let A be a square
matrix of order n, if the rank of matrix A is n-1, then the rank of the adjoint of matrix A is 1. For a square matrix, the transpose of the cofactor matrix In linear algebra, the adjugate or classical adjoint of a square matrix A, adj(A), is the transpose of its cofactor matrix.[1][2] It is occasionally known as adjunct matrix,[3][4] or "adjoint",[5] though that
normally refers to a different concept, the adjoint operator which for a matrix is the conjugate transpose. The product of a matrix with its adjugate gives a diagonal matrix (entries not on the main diagonal are zero) whose diagonal entries are the determinant of the original matrix: A adj (A) = det (A) I , {\displaystyle \mathbf {A} \operatorname
{adj} (\mathbf {A})=\det(\mathbf {A})\mathbf {I} ,} where I is the identity matrix of the same size as A. Consequently, the multiplicative inverse of an invertible matrix can be found by dividing its adjugate by its determinant. The adjugate of A is the transpose of the cofactor matrix C of A, adj (A) = C T . {\displaystyle \operatorname {adj}
(\mathbf {A})=\mathbf {C} ^{\mathsf {T}}.} In more detail, suppose R is a (unital) commutative ring and A is an n × n matrix with entries from R. The (i, j)-minor of A, denoted Mij, is the determinant of the (n − 1) × (n − 1) matrix that results from deleting row i and column j of A. The cofactor matrix of A is the n × n matrix C whose (i, j) entry is
the (i, j) cofactor of A, which is the (i, j)-minor times a sign factor: C = ((− 1) i + j M i j) 1 ≤ i , j ≤ n . {\displaystyle \mathbf {C} =\left((-1)^{i+j}\mathbf {M} _{ij}\right)_{1\leq i,j\leq n}.} The adjugate of A is the transpose of C, that is, the n × n matrix whose (i, j) entry is the (j,i) cofactor of A, adj (A) = C T = ((− 1) i + j M j i) 1 ≤ i , j ≤ n .
{\displaystyle \operatorname {adj} (\mathbf {A})=\mathbf {C} ^{\mathsf {T}}=\left((-1)^{i+j}\mathbf {M} _{ji}\right)_{1\leq i,j\leq n}.} The adjugate is defined so that the product of A with its adjugate yields a diagonal matrix whose diagonal entries are the determinant det(A). That is, A adj (A) = adj (A) A = det (A) I , {\displaystyle \mathbf
{A} \operatorname {adj} (\mathbf {A})=\operatorname {adj} (\mathbf {A})\mathbf {A} =\det(\mathbf {A})\mathbf {I} ,} where I is the n × n identity matrix. This is a consequence of the Laplace expansion of the determinant. The above formula implies one of the fundamental results in matrix algebra, that A is invertible if and only if det(A) is an
invertible element of R. When this holds, the equation above yields adj (A) = det (A) A − 1 , A − 1 = det (A) − 1 adj (A) . {\displaystyle {\begin{aligned}\operatorname {adj} (\mathbf {A})&=\det(\mathbf {A})\mathbf {A} ^{-1},\\\mathbf {A} ^{-1}&=\det(\mathbf {A})^{-1}\operatorname {adj} (\mathbf {A}).\end{aligned}}} Since the
determinant of a 0 × 0 matrix is 1, the adjugate of any 1 × 1 matrix (complex scalar) is I = [1] {\displaystyle \mathbf {I} ={\begin{bmatrix}1\end{bmatrix}}} . Observe that A adj (A) = adj (A) A = (det A) I . {\displaystyle \mathbf {A} \operatorname {adj} (\mathbf {A})=\operatorname {adj} (\mathbf {A})\mathbf {A} =(\det \mathbf {A}
)\mathbf {I} .} The adjugate of the 2 × 2 matrix A = [a b c d] {\displaystyle \mathbf {A} ={\begin{bmatrix}a&b\\c&d\end{bmatrix}}} is adj (A) = [d − b − c a] . {\displaystyle \operatorname {adj} (\mathbf {A})={\begin{bmatrix}d&-b\\-c&a\end{bmatrix}}.} By direct computation, A adj (A) = [a d − b c 0 0 a d − b c] = (det A) I .
{\displaystyle \mathbf {A} \operatorname {adj} (\mathbf {A})={\begin{bmatrix}ad-bc&0\\0&ad-bc\end{bmatrix}}=(\det \mathbf {A})\mathbf {I} .} In this case, it is also true that det(adj(A)) = det(A) and hence that adj(adj(A)) = A. Consider a 3 × 3 matrix A = [a 1 a 2 a 3 b 1 b 2 b 3 c 1 c 2 c 3] . {\displaystyle \mathbf {A} =
{\begin{bmatrix}a_{1}&a_{2}&a_{3}\\b_{1}&b_{2}&b_{3}\\c_{1}&c_{2}&c_{3}\end{bmatrix}}.} Its cofactor matrix is C = [+ | b 2 b 3 c 2 c 3 | − | b 1 b 3 c 1 c 3 | + | b 1 b 2 c 1 c 2 | − | a 2 a 3 c 2 c 3 | + | a 1 a 3 c 1 c 3 | − | a 1 a 2 c 1 c 2 | + | a 2 a 3 b 2 b 3 | − | a 1 a 3 b 1 b 3 | + | a 1 a 2 b 1 b 2 |] , {\displaystyle \mathbf {C} =
{\begin{bmatrix}+{\begin{vmatrix}b_{2}&b_{3}\\c_{2}&c_{3}\end{vmatrix}}&-{\begin{vmatrix}b_{1}&b_{3}\\c_{1}&c_{3}\end{vmatrix}}&+{\begin{vmatrix}b_{1}&b_{2}\\c_{1}&c_{2}\end{vmatrix}}\\\\-{\begin{vmatrix}a_{2}&a_{3}\\c_{2}&c_{3}\end{vmatrix}}&+{\begin{vmatrix}a_{1}&a_{3}\\c_{1}&c_{3}\end{vmatrix}}&-
{\begin{vmatrix}a_{1}&a_{2}\\c_{1}&c_{2}\end{vmatrix}}\\\\+{\begin{vmatrix}a_{2}&a_{3}\\b_{2}&b_{3}\end{vmatrix}}&-{\begin{vmatrix}a_{1}&a_{3}\\b_{1}&b_{3}\end{vmatrix}}&+{\begin{vmatrix}a_{1}&a_{2}\\b_{1}&b_{2}\end{vmatrix}}\end{bmatrix}},} where | a b c d | = det [a b c d] . {\displaystyle
{\begin{vmatrix}a&b\\c&d\end{vmatrix}}=\det \!{\begin{bmatrix}a&b\\c&d\end{bmatrix}}.} Its adjugate is the transpose of its cofactor matrix, adj (A) = C T = [+ | b 2 b 3 c 2 c 3 | − | a 2 a 3 c 2 c 3 | + | a 2 a 3 b 2 b 3 | − | b 1 b 3 c 1 c 3 | + | a 1 a 3 c 1 c 3 | − | a 1 a 3 b 1 b 3 | + | b 1 b 2 c 1 c 2 | − | a 1 a 2 c 1 c 2 | + | a 1 a 2 b 1 b 2 |] .
{\displaystyle \operatorname {adj} (\mathbf {A})=\mathbf {C} ^{\mathsf {T}}={\begin{bmatrix}+{\begin{vmatrix}b_{2}&b_{3}\\c_{2}&c_{3}\end{vmatrix}}&-{\begin{vmatrix}a_{2}&a_{3}\\c_{2}&c_{3}\end{vmatrix}}&+{\begin{vmatrix}a_{2}&a_{3}\\b_{2}&b_{3}\end{vmatrix}}\\&&\\-
{\begin{vmatrix}b_{1}&b_{3}\\c_{1}&c_{3}\end{vmatrix}}&+{\begin{vmatrix}a_{1}&a_{3}\\c_{1}&c_{3}\end{vmatrix}}&-{\begin{vmatrix}a_{1}&a_{3}\\b_{1}&b_{3}\end{vmatrix}}\\&&\\+{\begin{vmatrix}b_{1}&b_{2}\\c_{1}&c_{2}\end{vmatrix}}&-{\begin{vmatrix}a_{1}&a_{2}\\c_{1}&c_{2}\end{vmatrix}}&+
{\begin{vmatrix}a_{1}&a_{2}\\b_{1}&b_{2}\end{vmatrix}}\end{bmatrix}}.} As a specific example, we have adj [− 3 2 − 5 − 1 0 − 2 3 − 4 1] = [− 8 18 − 4 − 5 12 − 1 4 − 6 2] . {\displaystyle \operatorname {adj} \!{\begin{bmatrix}-3&2&-5\\-1&0&-2\\3&-4&1\end{bmatrix}}={\begin{bmatrix}-8&18&-4\\-5&12&-1\\4&-6&2\end{bmatrix}}.} It
is easy to check the adjugate is the inverse times the determinant, −6. The −1 in the second row, third column of the adjugate was computed as follows. The (2,3) entry of the adjugate is the (3,2) cofactor of A. This cofactor is computed using the submatrix obtained by deleting the third row and second column of the original matrix A, [− 3 − 5 − 1 −
2] . {\displaystyle {\begin{bmatrix}-3&-5\\-1&-2\end{bmatrix}}.} The (3,2) cofactor is a sign times the determinant of this submatrix: (− 1) 3 + 2 det [− 3 − 5 − 1 − 2] = − (− 3 ⋅ − 2 − − 5 ⋅ − 1) = − 1 , {\displaystyle (-1)^{3+2}\operatorname {det} \!{\begin{bmatrix}-3&-5\\-1&-2\end{bmatrix}}=-(-3\cdot -2--5\cdot -1)=-1,} and this is the (2,3)
entry of the adjugate. For any n × n matrix A, elementary computations show that adjugates have the following properties: adj (I) = I {\displaystyle \operatorname {adj} (\mathbf {I})=\mathbf {I} } , where I {\displaystyle \mathbf {I} } is the identity matrix. adj (0) = 0 {\displaystyle \operatorname {adj} (\mathbf {0})=\mathbf {0} } , where 0
{\displaystyle \mathbf {0} } is the zero matrix, except that if n = 1 {\displaystyle n=1} then adj (0) = I {\displaystyle \operatorname {adj} (\mathbf {0})=\mathbf {I} } . adj (c A) = c n − 1 adj (A) {\displaystyle \operatorname {adj} (c\mathbf {A})=c^{n-1}\operatorname {adj} (\mathbf {A})} for any scalar c. adj (A T) = adj (A) T
{\displaystyle \operatorname {adj} (\mathbf {A} ^{\mathsf {T}})=\operatorname {adj} (\mathbf {A})^{\mathsf {T}}} . det (adj (A)) = (det A) n − 1 {\displaystyle \det(\operatorname {adj} (\mathbf {A}))=(\det \mathbf {A})^{n-1}} . If A is invertible, then adj (A) = (det A) A − 1 {\displaystyle \operatorname {adj} (\mathbf {A})=(\det
\mathbf {A})\mathbf {A} ^{-1}} . It follows that: adj(A) is invertible with inverse (det A)−1A. adj(A−1) = adj(A)−1. adj(A) is entrywise polynomial in A. In particular, over the real or complex numbers, the adjugate is a smooth function of the entries of A. Over the complex numbers, adj (A ¯) = adj (A) ¯ {\displaystyle \operatorname {adj}
({\overline {\mathbf {A} }})={\overline {\operatorname {adj} (\mathbf {A})}}} , where the bar denotes complex conjugation. adj (A ∗) = adj (A) ∗ {\displaystyle \operatorname {adj} (\mathbf {A} ^{*})=\operatorname {adj} (\mathbf {A})^{*}} , where the asterisk denotes conjugate transpose. Suppose that B is another n × n matrix. Then adj
(A B) = adj (B) adj (A) . {\displaystyle \operatorname {adj} (\mathbf {AB})=\operatorname {adj} (\mathbf {B})\operatorname {adj} (\mathbf {A}).} This can be proved in three ways. One way, valid for any commutative ring, is a direct computation using the Cauchy–Binet formula. The second way, valid for the real or complex numbers, is to
first observe that for invertible matrices A and B, adj (B) adj (A) = (det B) B − 1 (det A) A − 1 = (det A B) (A B) − 1 = adj (A B) . {\displaystyle \operatorname {adj} (\mathbf {B})\operatorname {adj} (\mathbf {A})=(\det \mathbf {B})\mathbf {B} ^{-1}(\det \mathbf {A})\mathbf {A} ^{-1}=(\det \mathbf {AB})(\mathbf {AB}
)^{-1}=\operatorname {adj} (\mathbf {AB}).} Because every non-invertible matrix is the limit of invertible matrices, continuity of the adjugate then implies that the formula remains true when one of A or B is not invertible. A corollary of the previous formula is that, for any non-negative integer k, adj (A k) = adj (A) k . {\displaystyle
\operatorname {adj} (\mathbf {A} ^{k})=\operatorname {adj} (\mathbf {A})^{k}.} If A is invertible, then the above formula also holds for negative k. From the identity (A + B) adj (A + B) B = det (A + B) B = B adj (A + B) (A + B) , {\displaystyle (\mathbf {A} +\mathbf {B})\operatorname {adj} (\mathbf {A} +\mathbf {B})\mathbf {B}
=\det(\mathbf {A} +\mathbf {B})\mathbf {B} =\mathbf {B} \operatorname {adj} (\mathbf {A} +\mathbf {B})(\mathbf {A} +\mathbf {B}),} we deduce A adj (A + B) B = B adj (A + B) A . {\displaystyle \mathbf {A} \operatorname {adj} (\mathbf {A} +\mathbf {B})\mathbf {B} =\mathbf {B} \operatorname {adj} (\mathbf {A} +\mathbf {B}
)\mathbf {A} .} Suppose that A commutes with B. Multiplying the identity AB = BA on the left and right by adj(A) proves that det (A) adj (A) B = det (A) B adj (A) . {\displaystyle \det(\mathbf {A})\operatorname {adj} (\mathbf {A})\mathbf {B} =\det(\mathbf {A})\mathbf {B} \operatorname {adj} (\mathbf {A}).} If A is invertible, this implies
that adj(A) also commutes with B. Over the real or complex numbers, continuity implies that adj(A) commutes with B even when A is not invertible. Finally, there is a more general proof than the second proof, which only requires that an n × n matrix has entries over a field with at least 2n + 1 elements (e.g. a 5 × 5 matrix over the integers modulo
11). det(A+tI) is a polynomial in t with degree at most n, so it has at most n roots. Note that the ijth entry of adj((A+tI)(B)) is a polynomial of at most order n, and likewise for adj(A+tI)adj(B). These two polynomials at the ijth entry agree on at least n + 1 points, as we have at least n + 1 elements of the field where A+tI is invertible, and we have
proven the identity for invertible matrices. Polynomials of degree n which agree on n + 1 points must be identical (subtract them from each other and you have n + 1 roots for a polynomial of degree at most n – a contradiction unless their difference is identically zero). As the two polynomials are identical, they take the same value for every value of t.
Thus, they take the same value when t = 0. Using the above properties and other elementary computations, it is straightforward to show that if A has one of the following properties, then adjA does as well: upper triangular, lower triangular, diagonal, orthogonal, unitary, symmetric, Hermitian, normal. If A is skew-symmetric, then adj(A) is skew-
symmetric for even n and symmetric for odd n. Similarly, if A is skew-Hermitian, then adj(A) is skew-Hermitian for even n and Hermitian for odd n. If A is invertible, then, as noted above, there is a formula for adj(A) in terms of the determinant and inverse of A. When A is not invertible, the adjugate satisfies different but closely related formulas. If
rk(A) ≤ n − 2, then adj(A) = 0. If rk(A) = n − 1, then rk(adj(A)) = 1. (Some minor is non-zero, so adj(A) is non-zero and hence has rank at least one; the identity adj(A)A = 0 implies that the dimension of the nullspace of adj(A) is at least n − 1, so its rank is at most one.) It follows that adj(A) = αxyT, where α is a scalar and x and y are vectors such that
Ax = 0 and AT y = 0. See also: Cramer's rule Partition A into column vectors: A = [a 1 ⋯ a n] . {\displaystyle \mathbf {A} ={\begin{bmatrix}\mathbf {a} _{1}&\cdots &\mathbf {a} _{n}\end{bmatrix}}.} Let b be a column vector of size n. Fix 1 ≤ i ≤ n and consider the matrix formed by replacing column i of A by b: (A ← i b)   = def   [a 1 ⋯ a i − 1
b a i + 1 ⋯ a n] . {\displaystyle (\mathbf {A} {\stackrel {i}{\leftarrow }}\mathbf {b})\ {\stackrel {\text{def}}{=}}\ {\begin{bmatrix}\mathbf {a} _{1}&\cdots &\mathbf {a} _{i-1}&\mathbf {b} &\mathbf {a} _{i+1}&\cdots &\mathbf {a} _{n}\end{bmatrix}}.} Laplace expand the determinant of this matrix along column i. The result is entry i of
the product adj(A)b. Collecting these determinants for the different possible i yields an equality of column vectors (det (A ← i b)) i = 1 n = adj (A) b . {\displaystyle \left(\det(\mathbf {A} {\stackrel {i}{\leftarrow }}\mathbf {b})\right)_{i=1}^{n}=\operatorname {adj} (\mathbf {A})\mathbf {b} .} This formula has the following concrete
consequence. Consider the linear system of equations A x = b . {\displaystyle \mathbf {A} \mathbf {x} =\mathbf {b} .} Assume that A is non-singular. Multiplying this system on the left by adj(A) and dividing by the determinant yields x = adj (A) b det A . {\displaystyle \mathbf {x} ={\frac {\operatorname {adj} (\mathbf {A})\mathbf {b} }{\det
\mathbf {A} }}.} Applying the previous formula to this situation yields Cramer's rule, x i = det (A ← i b) det A , {\displaystyle x_{i}={\frac {\det(\mathbf {A} {\stackrel {i}{\leftarrow }}\mathbf {b})}{\det \mathbf {A} }},} where xi is the ith entry of x. Let the characteristic polynomial of A be p (s) = det (s I − A) = ∑ i = 0 n p i s i ∈ R [s] .
{\displaystyle p(s)=\det(s\mathbf {I} -\mathbf {A})=\sum _{i=0}^{n}p_{i}s^{i}\in R[s].} The first divided difference of p is a symmetric polynomial of degree n − 1, Δ p (s , t) = p (s) − p (t) s − t = ∑ 0 ≤ j + k < n p j + k + 1 s j t k ∈ R [s , t] . {\displaystyle \Delta p(s,t)={\frac {p(s)-p(t)}{s-t}}=\sum _{0\leq j+k